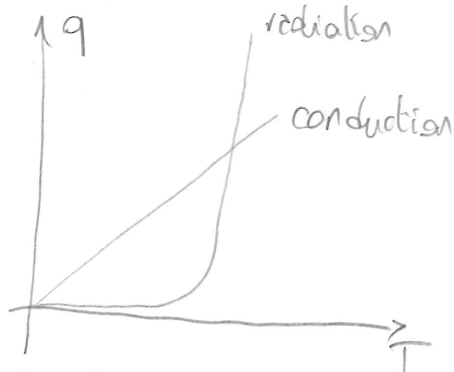
Fundamentals of heat radiation

General introduction

A good start for these notes could be quoting Wikipedia's definition of heat: "in thermodynamics, heat is energy in transfer to or from a thermodynamic system, by mechanisms other than thermodynamic work or transfer of matter".
To be clearer, thermodynamic work is energy transferred with mechanisms that spontaneously exert macroscopic forces, e.g., pressure, gravity, electromagnetism.
On the other hand, examples of transfer of matter, or "mass transfer", could regard chemistry, e.g., reactions, thermodynamic gradients, separations.
Finally, heat is simply "everything that does not fit in the previous definitions".
There are two very fundamental mechanisms supporting heat transfer: conduction, and radiation. [Actually, there is a third mechanism often invoked, which is convection, but this is a special case of conduction, where there is macroscopic movement of molecules outside of an imposed temperature gradient.]
Conduction is ruled by the temperatures of the media involved in the transfer, or, more accurately, on their difference: if temperature is constant, i.e., it is the same in all the media of interest, then there is no heat conduction.
On the other hand, heat radiation is in itself independent of temperature. [This statement is quite confusing/inexact, but Planck in [1] provides an idea with a nice concept example: imagine to concentrate the solar rays at a focus by passing them through a converging lens made of ice: the latter would remain at constant (freezing) temperature, but the body in the focus would catch fire.]

A more precise statement can be attempted by using an analogy. Heat conduction resembles electrical conduction, induced by a difference of potential. In heat, the role of potential is played by temperature. In the case of heat radiation, temperature does not play such a direct role. Nevertheless, the fact that a body releases heat by radiation is induced by the fact that its temperature is (very) high. The familiar refrain about heat radiation is that it does matter at high temperatures. Indeed, if conduction is proportional to ΔT, then it can be seen that a body radiates heat depending on T^4 (this result is known as "Stefan-Boltzmann law").

Also, conduction requires heat to go from one medium to another. Heat radiation, on the other hand, does not require a supporting medium to be propagated. Think about the Sun: it is 149.6×10^6 km far from Earth, in outer space (basically, vacuum), and yet it provides heat to us. Of course, behind its strong emission lies a very high surface temperature, which is known to be about 5777 K.



Qualitative idea of dependence of conduction and radiation of heat q vs temperature T

Characterizing heat radiation requires dealing with two branches of physics, namely, (i) electromagnetics and (ii) quantum physics. Electromagnetics does matter because, after being emitted, a "heat ray" is indistinguishable from a "light ray". Then, it is subject to the laws of optics, such as Snell, Fresnel, and so on. Also, rays can be characterized by their frequency ν, their energy E, their wavelength λ, their speed c, and so on and so forth.

Energy and frequency are related by the de Broglie relation

$$E = h\nu = \hbar\omega, \qquad (1)$$

where $\omega = 2\pi\nu$ is the angular frequency, h is the Planck's constant

$$h = 6.62607004 0 \times 10^{-34} \ \text{Js},$$

and $\hbar = h/(2\pi)$ is the "reduced Planck's constant".

Energy and frequency are "absolute" properties of the thermal wave, independent of media. Even if radiation of heat exists regardless of the presence of a medium, the properties of a wave might be affected by it. This is the case of the velocity of the wave. In vacuum, a wave (electromagnetic/thermal) propagates at speed c_0,

$$c_0 = 2.99792458 \ \text{m/s},$$

but in a material characterized by refractive index n (that is defined as the square root of the relative dielectric permittivity ε_r, hence it is a dimensionless number), we have speed c

$$c = \frac{c_0}{n}. \qquad (2)$$

Another concept broadly used is that of wavelength, λ. This is defined as

$$\lambda = \frac{c}{\nu}, \qquad (3)$$

hence it depends on the material. The wavelength λ is important both for "tradition", since in optics it is very used [possibly because in optics it is the quantity directly measured when trying to characterize experimentally the color of a ray/wave [1, p. 5]], and because λ is directly comparable with the space scales involved in our problems [the fact that, e.g., in nanostructures, one can intuitively and qualitatively imagine which is the operation of the device by studying if its geometrical details are large or small, compared to λ].

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This is half of the story, since the electromagnetic model cannot explain what's behind heat radiation, e.g., the radiative properties of gases or the idea/concept of black body, requiring also the development of a quantum model, which will be addressed in the following sections.

Black body and its characteristics

One of the central concepts in our presentation is that of a black body. Its definition is apparently simple, but actually it hides a lot of insight.

"A black body is one that allows all incident radiation and internally absorbs all of it."

Optically, given some wave impinging on some body, it might experiene one of the following phenomena: being reflected by the object, being transmitted through it, or being absorbed. This is summarized by the equation

$$\rho + \tau + \alpha = 1 \qquad (4)$$

In order to make (4) a bit more quantitative, consider a plane wave incident on an interface between two media with refractive indexes n_1, n_2. According to the Fresnel equations (we are considering normal incidence), we have that the reflection and transmission coefficients are:

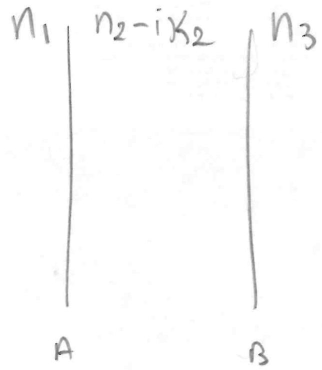
$$r = \frac{n_1 - n_2}{n_1 + n_2} \; ; \quad t = \frac{2\sqrt{n_1 n_2}}{n_1 + n_2} \, . \qquad (5)$$

The quantities ρ, τ in (4) are simply, in this case,

$$\rho = |r|^2$$
$$\tau = |t|^2 , \qquad (6)$$

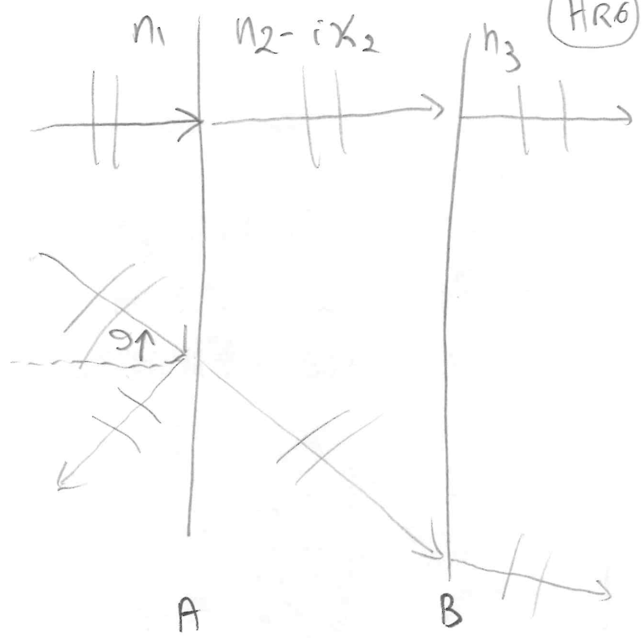
i.e., coefficients quantifying the <u>power</u> reflected by, and transmitted through, the interface.

Now: imagine to have something a bit more complex, i.e., a dielectric slab, where the internal layer is lossy [we need to have this additional complication, to introduce α, because assuming the first or last layers as lossy leads to theoretical complications: being them half-infinitely extended, to have a finite power at interface A means having an ∞ power impinging in the structure; probably this is less critical in the ending section...]. With additional complications, one can evaluate ρ and τ. Then, what is <u>absorbed</u> by the slab, α, is



$$\alpha = 1 - \rho - \tau \qquad (7).$$

Observe that, in general, ρ, τ and α depend on the angle θ of incidence. Considering the figure on the right, ones intuitively understands that, if θ is larger, than the wave stays "longer" in the lossy layer, so α is larger. Also, it is known that, even in lossless cases, ρ and τ exhibit a strong dependence on θ.



This is even more complex in 3D cases, where one has also the azimuth angle φ! [being the slab a planar structure, one can rotate the reference system to avoid dealing with φ, but in general it is not possible ...].

Having understood the meaning of the actors of (4), it is possible to provide a "more numerical" definition of a blackbody, as a body where

$$\begin{cases} \rho = 0 \\ \tau = 0 \end{cases} \qquad (8)$$

This could clarify why it is called "black" body: if it reflects, then we can see its color; if it transmits, then part of the radiation reaches the next obstacle, is back-transmitted, and we see through it: it would be partially transparent!

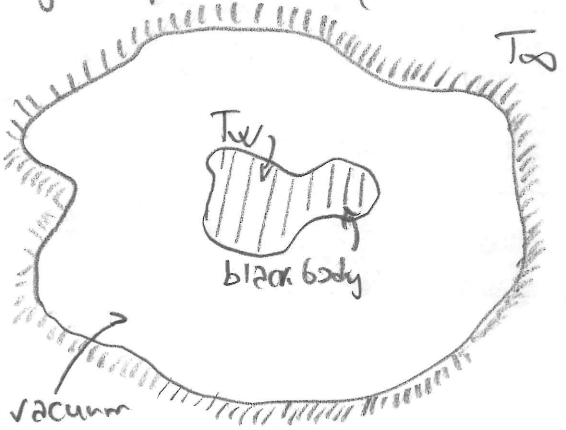
A note on λ: apart from ideal cases, these properties depend on λ (hence on E, ν ...). Consider again the slab, where l_{AB}

identifies the thickness of the central layer. If $\lambda \gg l_{AB}$, then the central layer is almost invisible "from the wave's perspective", and the slab behaves almost as an interface n_1/n_3. If $\lambda \ll l_{AB}$, then all the incident power overcoming the first interface gets absorbed, and this behaves basically as a single (left) interface, the second one is invisible. If $\lambda \approx l_{AB}$, we have the "usual" slab behaviour. Hence, ρ, τ, α are reasonably depending also on λ. Also, materials are dispersive, so $n_1 = n_1(\lambda)$, etc. In this view, one object could behave (more or less) as a blackbody, just in a certain λ range.

An "ideal black body" absorbs all the radiation, at whatever λ, at whatever θ or φ: there can be no body that absorbs more radiation than a blackbody in whatever λ, θ, φ!
In this view, this "conceptual definition" of black body serves as benchmark against which real surfaces are compared.

This definition of black body has been formulated under the assumption of having, from somewhere, a wave that reaches the black body (and gets absorbed by it). But, even if $\rho = 0$ and $\tau = 0$, another property of the black body is that it emits radiation, i.e., it generates heat rays. We are going deeper in this, in the next section.

Black body as an emitter
Let's consider the scenario of the figure on the right.



Let's consider a chamber with temperature fixed to T_∞. (HR8)

Inside it, we have a blackbody, small compared to the chamber, at initial temperature $T_w > T_\infty$. Let the chamber be internally evacuated, i.e., there is no medium between walls and black body: no conduction / convection can be present. In this view, radiation is the only mechanism supporting heat transfer. Assume that the temperature of the walls of the cavity is fixed and cannot change.

At the initial time the black body is not at equilibrium with the cavity. To reach equilibrium, the body must release energy, by emission. This is reasonable: even just because the body has a non-zero temperature, it emits heat radiation [this will be also quantified later]. The body will release heat, and its temperature will decrease, until equilibrium is reached. Equilibrium is that situation for which the net exchange of energy with the surroundings is 0. In this view, because energy is exchanged only in terms of radiation, it means that the body emits exactly the same radiation that it absorbes from the surroundings. Assuming that the cavity has the same emission properties of the blackbody, i.e., it is a blackbody too, then, @ equilibrium, the blackbody temperature will be T_∞.

Finally: because emission = absorption, but we know that the blackbody is, by definition, the best possible absorber, then it is also the best emitter possible!!

Note that the previous considerations are independent of the position and/or orientation of the black body inside the cavity, meaning that the radiation film is uniform and isotropic.

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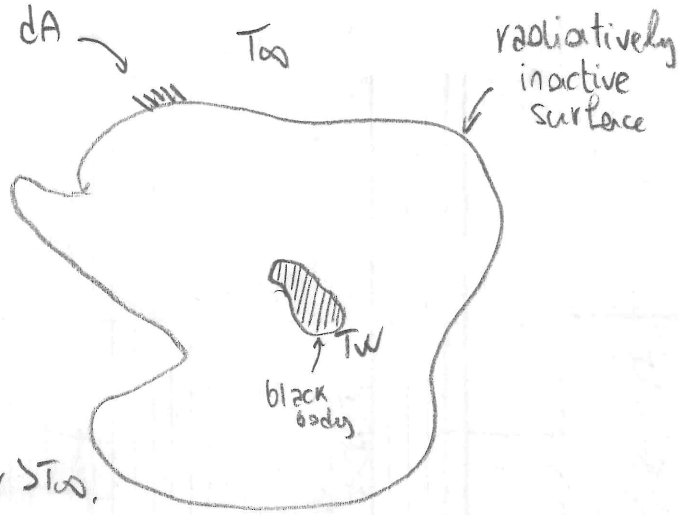


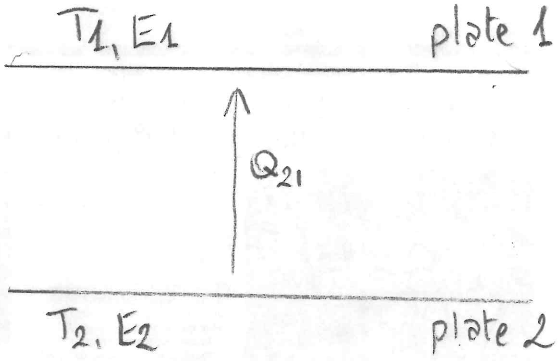
Consider now a different experiment: again a blackbody in an evacuated cavity, but such that only a small area, dA, is radiatively active. Again, the black body is at a temperature $T_W > T_\infty$.

At equilibrium, the black body will reach T_∞. But, in this case, the heat exchange may take place only with dA. In this view, radiation will be emitted/absorbed only in the direction of dA. So, because this is true for whatever dA, the black body has maximum emission and/or absorption in all directions.

The same considerations can be applied to a cavity designed to emit/absorb only at wavelength λ, and in a small interval $d\lambda$ about λ. In this case, the heat exchange might take place only in $d\lambda$, but, still, the blackbody goes from T_W to T_∞. Hence, at every wavelength, the black body is a perfect emitter. [note that, however, the black body emission depends on λ, as we are going to find soon!].

In all the experiments we have "performed" so far, in some way, the result was always the same: from T_W, our body reached T_∞. In this view, it is clear that the characteristics of the shape of body and/or cavity do not affect emission/absorption: the parameter that does really matter is <u>temperature</u>!

Also, it is possible to state that the radiation strength increases with temperature.



To this aim, consider two plates, one at temperature T_1 radiating with strength E_1, the other with T_2 and E_2.

Assume $T_1 > T_2$, but $E_1 < E_2$, i.e., the opposite of our intuition.

Because we assume heat transfer to be ruled by radiation, the heat transfer from plate 2 to plate 1 is

$$Q_{21} = E_2 - E_1 > 0. \qquad (9)$$

So, energy is moving from plate 2 to plate 1. But this is absurd! The law of thermodynamics, indeed, state that positive transfer of energy must take place from bodies at high temperature towards bodies at lower T. Hence, $E \propto \frac{1}{T}$ is absurd!

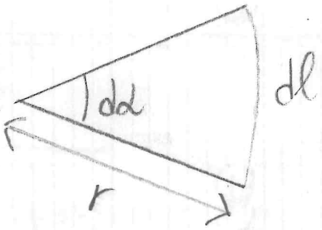
In other words, this demonstrates, through a "reductio ad absurdum", that the radiation strength is proportional to T!

Quantitative aspects of the black body behavior

In the previous section we have discussed, qualitatively, what a black body is. Also, in the very end, we have mentioned a certain quantity "E_i" but we have not perfectly clarified what it is. The scope of this section is to provide a more quantitative description of the black body behavior.

Preliminary definitions of solid geometry

What is an angle? Imagine to have a segment of a circle of (in this case, infinitesimal) length dl, radius r. Then, the corresponding angle, $d\alpha$, is:

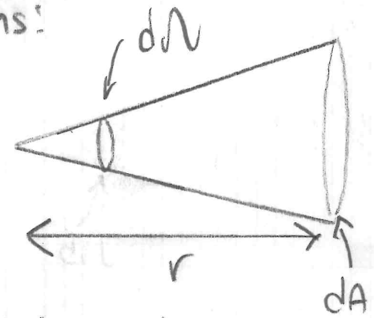


$$d\alpha = \frac{dl}{r} \quad (10)$$

In fact, this is exactly the definition of the angle based on the concept of "radian" [see Wikipedia, "radian"].

This idea can be extended to solid geometry, leading to the definition of "solid angle" based on steradians:

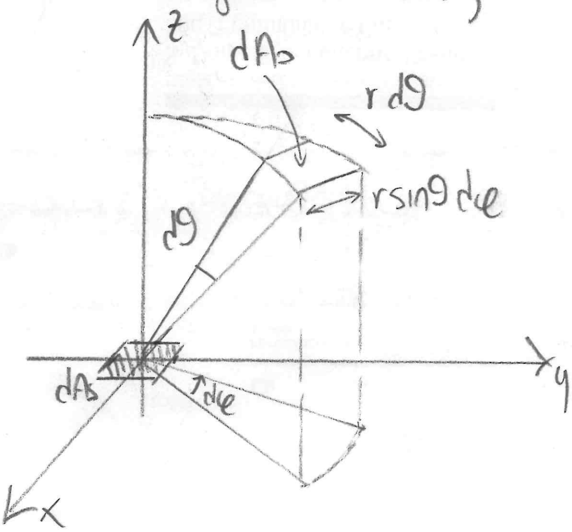
$$d\Omega = \frac{dA}{r^2}, \quad (11)$$



where dA is a portion of spherical surface, in this case, infinitesimal. Playing with infinitesimal areas is nice, because it allows to "confuse" planar and spherical areas. By this way, $d\Omega$ is the solid angle measured in steradians, i.e., the amplitude of the angle subtending the portion of spherical surface that, deployed on a plane, is compared to the square of the sphere radius (1 steradian is such that this area is equal to r^2).

Setting up the heat transfer problem

The definition of solid angle implies the presence of two actors: an origin, to which we refer the solid angle, and the area subtended by it. This is just about solid geometry. On the other hand, in radiative heat transfer, we need two actors as well: a "giver", i.e., a surface emitting radiation, and a "taker", i.e., a surface intercepting radiation, to be later absorbed. Solid angles are very useful to describe this scenario.



The "source" area is dA_s, from which the radiation is emitted. The solid angle is referred to it.

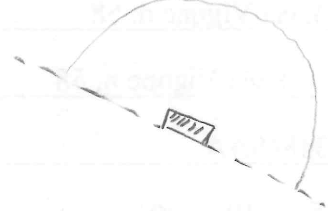
Then, the "observation" area, dA_o, is defined as the spherical area element at a distance r from dA_s, achieved by varying infinitesimally the spherical angles, with increments $d\theta$ and $d\varphi$. In this view, by inspection, one sees that the edges of the area element are $r\,d\theta$ and $r\sin\theta\,d\varphi$. Therefore, the area is the product of the two:

$$dA_o = (r\sin\theta\,d\varphi)(r\,d\theta) = r^2\sin\theta\,d\theta\,d\varphi . \qquad (12)$$

By recalling (11), it is now possible to provide an operative expression of the solid angle $d\Omega_o$ subtending dA_o, as:

$$d\Omega_o = \frac{dA_o}{r^2} = \frac{r^2\sin\theta\,d\theta\,d\varphi}{r^2} = \sin\theta\,d\theta\,d\varphi . \qquad (13)$$

As already mentioned, our interest is in radiation heat transfer between surfaces, of finite area. The source surface emits radiation, and it spreads in all directions, identifying a hemisphere. In this view, a preliminary exercise is evaluating the solid angle of such hemisphere. This is simply:



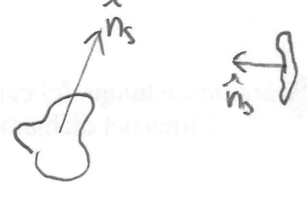
$$\Omega = \iint d\Omega_0 = \int_0^{2\pi}\int_0^{\frac{\pi}{2}} \sin\theta \, d\theta \, d\varphi =$$

$$= 2\pi \int_0^{\frac{\pi}{2}} \sin\theta \, d\theta = 2\pi \left[-\cos\theta\right]_0^{\frac{\pi}{2}} = 2\pi \text{ sr.} \quad (14)$$

First definition of spectral radiation intensity $I_{\lambda,e}$

In this section we are going to introduce a quantity called "spectral radiation intensity", $I_{\lambda,e}(\lambda, \theta, \varphi)$.

We are studying heat transfer. In conduction/convection, the sensitive quantity quantifying heat transfer is the heat flux q, whose units are $\frac{W}{m^2}$. Alternatively one can find Q, with units W (including the area). So: why are we introducing another quantity, and, how is it related to q? The answer to this doubt lies in the nature of radiative heat transfer. Indeed, radiation can come from whatever direction, and contain whatever spectral (i.e., wavelength) contribution: it depends on λ, θ, φ. Also, it is important to keep into account the directional orientation of one surface with respect to the other: if the surfaces are perfectly facing each other, then radiative transfer will be maximum. But, if instead the two surfaces are not "facing", i.e., their normals are not parallel,



then radiative transfer is lower. In this view, radiative transfer, and then $I_{\lambda,e}$, does not depend just on the area, but rather on the projected area, i.e., $\hat{n}_o \cdot \hat{n}_s = \cos\theta$.
Ultimately, it is possible to define the spectral radiation intensity as:

$$I_{\lambda,e}(\lambda,\theta,\varphi) = \frac{dQ}{dA_s \cos\theta \, d\Omega_o \, d\lambda}. \quad (15)$$

The subscript "λ,e" remarks that this is a spectral intensity (λ), and that we are focusing on emission (e). The units are:

$$\left[I_{\lambda,e}\right] = \frac{W}{m^2 \, \mu m \, sr}, \quad (16)$$

indeed, m^2 is given by the projected area $dA_s \cos\theta$. μm is used to indicate the units of wavelength λ, and sr is related to $d\Omega_o$.
In this view, $I_{\lambda,e}$ is the rate at which radiant energy is emitted by a surface, in the direction θ, per unit surface, per unit area normal to the surface, per unit solid angle $d\Omega_o$ about (θ,φ), in the unit wavelength interval $d\lambda$ about λ.
Even if not written explicitly, $I_{\lambda,e}$ depends also on the temperature T.

The spectral radiation intensity, in conjunction with our preliminary considerations on solid angles, are a convenient platform to set up a quantitative theory of black body radiation. Notice that, so far, the definition (15) for $I_{\lambda,e}$ is quite abstract, and unrelated to the concept of black body. As of this point we don't know exactly which is the expression of $I_{\lambda,b}$, i.e., the blackbody spectral radiation intensity, but we know some of its properties, such as, that it does not depend on θ or φ. This allows to do some preliminary calculations. For instance, it is already possible to define the spectral emissive power from a blackbody, $E_b(\lambda)$, as

$$E_{\lambda,b}(\lambda,T)=\int_0^{2\pi}\int_0^{\frac{\pi}{2}} I_{\lambda,b}\cos\theta\sin\theta\, d\theta\, d\varphi \quad . \quad (17)$$

More precisely, (17) is the spectral hemispherical emissive power of a black body. The fact that we are referring to a black body is emphasized by the subscript "b", rather than "e", which indicated a generic emitter. It is "spectral" because it is still a function of λ, since it has been integrated only in $d\Omega$.

By performing an integral also w.r.t. λ, it is possible to obtain the hemispherical total emissive power of the black body, $E_b(T)$, as

$$E_b(T)=\int_0^{\infty} E_{\lambda,b}(\lambda)\, d\lambda \qquad (18)$$

Equations (18) and (17) can be combined more explicitly, leading to:

$$E_b(T)=\int_0^{\infty}\int_0^{2\pi}\int_0^{\frac{\pi}{2}} I_{b,\lambda}(\lambda,T,\theta,\varphi)\cos\theta\sin\theta\, d\theta\, d\varphi\, d\lambda =$$

$$=\int_0^{\infty} I_{b,\lambda}(\lambda,T)\, d\lambda \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\cos\theta\sin\theta\, d\theta\, d\varphi = \qquad (19)$$

$$\triangleq I_b(T)\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\cos\theta\sin\theta\, d\theta\, d\varphi = 2\pi\, I_b(T)\int_0^{\frac{\pi}{2}}\cos\theta\sin\theta\, d\theta .$$

$$=2\pi\, I_b(T)\int_0^{\frac{\pi}{2}}\frac{1}{2}\sin(2\theta)\, d\theta = 2\pi\, I_b(T)\left[-\frac{1}{4}\cos(2\theta)\right]_0^{\frac{\pi}{2}} =$$

$$=2\pi\, I_b(T)\left(\frac{1}{2}\right)=\pi\, I_b(T). \qquad (20)$$

In (19), we have exploited the property, derived on the basis of the definition of a black body, that I_b is not depending on θ or φ.

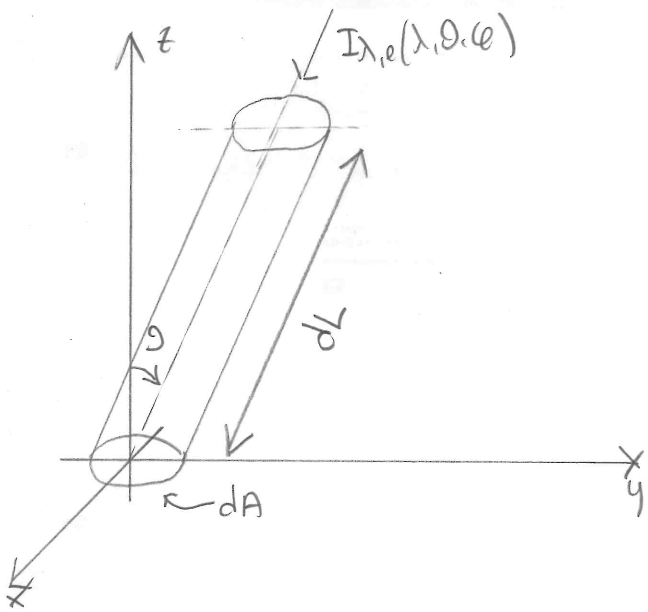
Then, we have implicitly defined

$$I_b(T) = \int_0^\infty I_{\lambda,b}(\lambda,T)\, d\lambda . \quad (21)$$

Apart from the few considerations applied in this section, we have almost no idea of how $I_{\lambda,b}(\lambda,T)$ behaves, or, in particular, which is its expression. In this view, the scope of the following sections is going to be its derivation.

Relating spectral radiation intensity and radiation energy density

For a generic emitter, we have previously reported the definition of spectral radiation intensity (15); this is now reported, isolating the radiation flux dQ as:



$$dQ = I_{\lambda,e}(\lambda,\vartheta,\varphi)\, dA \cos\vartheta \, d\Omega \, d\lambda , \quad (22)$$

where, now, we are indicating the area intercepting our radiation with $dA\cos\vartheta$. Radiation is in fact incident with an angle ϑ, so that its projection on the area element depends on $\cos\vartheta$.

The radiation flux dQ has units:

$$[dQ] = \frac{W}{m^2} . \quad (23)$$

Now: is it possible to relate (22) to an energy density? Well, the idea is based on recalling that energy is related to power times time. Therefore, an energy density is achieved by considering

$$dE_\lambda(\lambda,\vartheta,\varphi) = dQ\, dt = I_{\lambda,e}\, dA \cos\vartheta \, d\Omega \, d\lambda \, dt . \quad (24)$$

Introducing dt changes quite much our physics. Indeed now, rather than studying a power flowing through a surface, we have to investigate an energy density stored in a volume. Indeed, because of this dt, our power will overcome dA, continue propagating during dt, filling the volume shown in the previous picture, dV. This dV can be written as:

$$dV = dA\cos\theta\, dL, \quad (25)$$

[the $\cos\theta$ because the incidence is skew, hence the volume where energy is contained depends on the orientation w.r.t. the wave]

where dL is the length spanned by our wave, i.e.,

$$dL = c\,dt, \quad (26)$$

assuming that the wave travels at a speed c (material-dependent) in our volume.

So, we can define a radiation energy <u>density</u> (per unit volume) as:

$$du_\lambda(\lambda,\theta,\varphi) = \frac{dE_\lambda(\lambda,\theta,\varphi)}{dV} = \frac{I_{\lambda,e}(\lambda,\theta,\varphi)\,\cancel{dA\cos\theta}\,d\Omega\,d\lambda\,\cancel{dt}}{c\,\cancel{dA\cos\theta}\,\cancel{dt}} =$$

$$= \frac{I_{\lambda,e}(\lambda,\theta,\varphi)}{c}\,d\Omega\,d\lambda. \quad (27)$$

The final result, u_λ, is achieved by integrating such expression on λ and Ω. Starting from λ, we have, recalling the definition (21) [proposed for a black-body but still compatible with the present context]

$$du(\theta,\varphi) = \frac{1}{c}\int_0^\infty I_{\lambda,e}(\lambda,\theta,\varphi)\,d\lambda = \frac{1}{c}\,I_e(T,\theta,\varphi). \quad (28)$$

Ultimately, u can be achieved as:

$$u = \frac{1}{c}\int_0^{2\pi}\int_0^{\pi} I_e(T,\theta,\varphi)\sin\theta\,d\theta\,d\varphi = \frac{2\pi}{c}\int_0^{\pi} I_e(T,\theta)\sin\theta\,d\theta \quad (29)$$

This expression is valid for a generic emitter. If we focus on a black body, we achieve:

$$u_b = \frac{2\pi}{c} I_b(T) \int_0^{\pi} \sin\theta \, d\theta = \frac{4\pi}{c} I_b(T) . \quad (30)$$

We reached our scope, i.e., relating, for a black body, the spectral radiation intensity with another concept, related to an energy density. Actually, this is even valid, for a black body, λ by λ, so it is possible to state

$$u_{b,\lambda}(\lambda) = \frac{4\pi}{c} I_{b,\lambda}(T,\lambda) , \quad (31)$$

simply by integrating (27) first on θ, i.e., assuming to deal with a black body from the beginning.

Obtaining the radiation energy density for a black body

We have introduced the concept of radiation energy density for a very specific reason: basing on it, it is possible to achieve an explicit expression describing quantitatively a black body.

We are going to deal with quantities related to energy, hence, it is appropriate to renounce focusing on wavelengths. The two worlds are related, but with some attention: an infinitesimal increment of wavelength does not correspond directly to an infinitesimal change of frequency (which, instead, is directly proportional to E), because

$\nu = \frac{c}{\lambda}$; therefore,

$$d\nu = \frac{c}{\lambda^2} d\lambda . \quad (32)$$

All the quantities we investigate make sense if multiplied times their increment. Hence, when converting, e.g., from u_λ to u_ν, we have to recall that

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$$|I_\lambda \, d\lambda| = |I_\nu \, d\nu| , \qquad \boxed{\text{and NOT simply } I_\lambda = I_\nu \, !} \tag{33}$$

So, in this section our goal will be determining u_ν, in a black body. A "body" is something finite; if we want to quantify the energy density in it, we should evaluate all the possible states in our body, and understanding how many photons are in each of those states. Hence, our u_ν is:

$$u_\nu \, d\nu = \frac{\text{number of "oscillations" between } \nu \text{ and } \nu + d\nu}{\text{box volume}} \times \text{average energy per wave} \; d\nu$$

$$= \frac{1}{V} N(\nu) \, d\nu \times \bar{E} , \qquad (34)$$

where each of the "states" that a photon, a "wave", can assume, is characterized by its number of oscillations (this is going to be clear in a while).

Because our body has a finite extent, we have to work with a box. The shape of such box is not very relevant (see HR9: the black body characteristics, in terms of radiation, are independent of its shape, etc.). So, let's do our computations on a cubic box with side length L. The governing equation for standing waves is:

$$\frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} . \qquad (35)$$

Our scope is to achieve $N(\nu) \, d\nu$, i.e., the number of waves at a

prescribed frequency ν; we are going to see that u_ν, in Proof, (HR20)
is independent of ν.

Because the box is finite, we need to enforce boundary conditions describing the finiteness. For example, we can use:

$$\psi = 0 \quad \text{for} \quad x=0 \quad \text{and} \quad x=L$$

$$y=0 \quad \text{and} \quad y=L$$

$$z=0 \quad \text{and} \quad z=L;$$

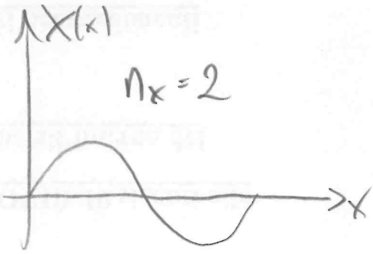
by this way, our solution ψ is separable:

$\psi = T(t)\,X(x)\,Y(y)\,Z(z)$,

and has the form

$$\psi_n = \left(A_n \cos(\omega_n t) + B_n \sin(\omega_n t)\right) \sin\left(n_x \frac{\pi}{L} x\right) \sin\left(n_y \frac{\pi}{L} y\right) \sin\left(n_z \frac{\pi}{L} z\right) \qquad (36)$$

It is now clear what we meant for "oscillations": for every triplet (n_x, n_y, n_z) we have a different state that photons can have, and, in particular, n_x indicates the number of oscillations of $X(x)$ along x. The same holds for y and z.



The circular frequency ω_n is actually related to n_x, n_y, n_z. Recalling that the wavenumber K satisfies

$$K^2 = K_x^2 + K_y^2 + K_z^2 = \left(\frac{\pi}{L}\right)^2 \left(n_x^2 + n_y^2 + n_z^2\right), \qquad (37)$$

since

$$K = \frac{\omega_n}{c} = \frac{2\pi\nu}{c}, \qquad (38)$$

$$\omega_n^2 = c^2 K^2 = \frac{\pi^2 c^2}{L^2}\left(n_x^2 + n_y^2 + n_z^2\right), \qquad (39)$$

Let's define p^2 as:

$$p^2 = n_x^2 + n_y^2 + n_z^2 . \quad (40)$$

Then, according to the previous formulas,

$$\omega_n^2 = \frac{\pi^2 c^2}{L^2} p^2 ,$$

or, with ν,

$$4\cancel{\pi^2}\nu^2 = \frac{\cancel{\pi^2} c^2}{L^2} p^2 \Rightarrow p^2 = \frac{4L^2}{c^2}\nu^2 \quad (41)$$

So,

$$p = \frac{2L}{c}\nu , \quad (42)$$

which allows to relate the increment of states, dp, to an increment of frequency, $d\nu$, by taking its differential as

$$dp = \frac{2L}{c} d\nu . \quad (43)$$

Before moving on, a brief summary: we are at a frequency ν, and we have a certain number of possible state configurations, because, for (42), all the states within the p-spherical surface with radius p are acceptable. These states can be counted by computing the volume of the sphere of radius p, which is, $\frac{4}{3}\pi p^3$. But this is not what we want: according to (34), we want the states lying in a shell between the spheres with radii p and $p + dp$. So, our scope is to compute $N(p)\,dp$ defined as:

$$N(p)\,dp = \frac{4}{3}\pi (p+dp)^3 - \frac{4}{3}\pi p^3 = \frac{4}{3}\pi\left(\cancel{p^3} + 3p^2 dp + 3p\,dp^2 + dp^3\right) - \cancel{\frac{4}{3}\pi p^3}$$

$$\approx 4\pi p^2 dp ,$$

where we are neglecting high-order infinitesimals dp^2, dp^3.
To sum up, the volume of the spherical shell is:

$$N_0(p)\,dp = 4\pi p^2\,dp. \quad (44)$$

Actually, we don't need this whole N_0, but just part of it, in particular, our scope is to compute

$$N(p)\,dp = 2 \times \frac{1}{8} \times N_0(p)\,dp. \quad (45)$$

because 2 field polarizations are possible, $\parallel$ and $\perp$

because only the first octant, with $n_x \geq 0$, $n_y \geq 0$, $n_z \geq 0$ is of interest; all the other states are the same, i.e., degenerate!

We are in shape enough to reach our point. In fact, starting from (44) and (45), we can substitute (42) and (43), leading to

$$N(p)\,dp = 2 \times \frac{1}{8} \times 4\pi p^2\,dp = \cancel{2} \times \cancel{\frac{1}{8}} \times 4\pi \times \frac{\cancel{4}L^2}{c^2}\nu^2\,\frac{2L}{c}\,d\nu =$$

$$= \frac{8\pi L^3 \nu^2}{c^3}\,d\nu. \quad (46)$$

$$= N(\nu)\,d\nu$$

This is what we needed. In fact, we can finally go back to (34), and write it as:

$$u_\nu(\nu)\,d\nu = \frac{1}{\cancel{V}}\,\frac{8\pi \cancel{L^3} \nu^2}{c^3}\,d\nu\,\bar{E} = \frac{8\pi \nu^2}{c^3}\,d\nu\,\bar{E}, \quad (47)$$

which provides a quite good point towards the determination of $u_\nu(\nu)$! =)

The Rayleigh-Jeans distribution

Equation (47) can be written as:

$$u_\nu(\nu) = \frac{8\pi\nu^2}{c^3}\,\bar{E}\,, \quad (48)$$

which actually indicates that we have done just half of the story: we still have not discussed $\bar{E}$, i.e., the average energy of the waves.

Our calculation of the "modes of the box" is more or less as treating each mode as a harmonic oscillator. Each mode has its own energy and, at the end of some transient, they all reach the same $\bar{E}$; this, at least, was the idea of Rayleigh and Jeans to try to evaluate such $\bar{E}$. So, by the equipartition theorem applied to a <u>classic</u> harmonic oscillator, it is recognized that energy has two contributions: potential and kinetic. Each of these contributions is known [see Wikipedia, "Equipartition theorem"] to contribute as:

$$\langle H_{kinetic} \rangle + \langle H_{potential} \rangle = \frac{1}{2} K_B T + \frac{1}{2} K_B T = K_B T, \quad (49)$$

where K_B indicates the Boltzmann constant,

$$K_B = 1.3806488 \times 10^{-23} \, \frac{J}{K}\,, \quad (50)$$

and, ultimately,

$$\bar{E} = K_B T\,,$$

so that (48) is, with (49), the Rayleigh-Jeans distribution:

$$u_\nu(\nu) = \frac{8\pi\nu^2}{c^3} K_B T \quad (51)$$

The Rayleigh-Jeans distribution is known to be quite correct at low energies (i.e., ν), but it has a tremendous problem: if we integrate it over ν, we have

$$\int_0^\infty u_T(\nu)\, d\nu \to \infty : \quad \text{it diverges!}$$

This is referred to as the "ultra violet catastrophe": this model is totally wrong at high energies! So, this does not work for all frequencies...

Planck's law

... And here is where Planck makes his glorious entrance. Planck had nothing to say about (48): all the problems, in his opinion, concerned how to evaluate $\bar{E}$. In particular, its calculation in the Rayleigh-Jeans version does not take into account the quantum nature of photons, namely, the photoelectric effect. Indeed, we know that the energy of each box mode is not unrelated to its frequency, due to the relation

$$E(\nu) = n h \nu, \quad (52)$$

n being the number of photons oscillating according to this mode. Then, Planck assumes that all the photons (hence, the modes) are at equilibrium. In this view, in order to determine the expectation value of the occupation of the various modes, Planck adopted a statistical mechanics approach, i.e., a probabilistic approach. According to Maxwell-Boltzmann statistics, the probability distribution over the energy levels of a particular mode is: [2004 Bruus, p. 26]

$$\mathbb{P}(n) = \frac{\exp\left(-\frac{E_n}{K_B T}\right)}{Z(T)}, \qquad (53)$$

where

$$Z = \sum_{K=0}^{\infty} \exp\left(-\frac{E_K}{K_B T}\right). \qquad (54)$$

The average energy can be computed, on the basis of (53), as the expectation value

$$\bar{E} = \bar{E}_\nu = \sum_{n=0}^{\infty} E_n \mathbb{P}(n) = \frac{\sum_{n=0}^{\infty} E_n \exp\left(-\frac{E_n}{K_B T}\right)}{\sum_{K=0}^{\infty} \exp\left(-\frac{E_K}{K_B T}\right)} =$$

$$= \frac{\sum_{n=0}^{\infty} nh\nu \exp\left(-\frac{nh\nu}{K_B T}\right)}{\sum_{K=0}^{\infty} \exp\left(-\frac{Kh\nu}{K_B T}\right)}. \qquad (55)$$

Evaluating (55) is much simpler than what one may imagine. Indeed, by applying the change of variables

$$x = \exp\left(-\frac{h\nu}{K_B T}\right),$$

it becomes

$$\bar{E}_\nu = h\nu \frac{\sum_{n=0}^{\infty} n x^n}{\sum_{n=0}^{\infty} x^n} = h\nu \frac{x + 2x^2 + 3x^3 + \dots}{1 + x + x^2 + \dots} = \frac{x(1 + 2x + 3x^2 + \dots)}{1 + x + x^2 + \dots}$$

Recalling that

$$\frac{1}{1-x} = 1 + x + x^2 + \dots, \qquad \frac{1}{1-x^2} = 1 + 2x + 3x^2 + \dots,$$

we have

$$\bar{E}_\nu = h\nu \frac{\frac{x}{(1-x)^{\cancel{2}}}}{\cancel{\frac{1}{1-x}}} = \frac{h\nu x}{1-x} = \frac{h\nu}{x^{-1}-1} =$$

$$= \frac{h\nu}{\exp\left(\frac{h\nu}{k_B T}\right)-1} . \quad (56)$$

This is what we aimed for! And, notice, it goes back, for $\nu \to 0$, to the classical limit! In fact, for $\nu \to 0$,

$$\exp\left(\frac{h\nu}{k_B T}\right) \simeq 1 + \frac{h\nu}{k_B T},$$

so that

$$E_\nu \approx \frac{h\nu}{\cancel{1} + \frac{h\nu}{k_B T} - \cancel{1}} \longrightarrow k_B T,$$ as found by Rayleigh and Jeans!

Ultimately, we have:

$$u_\nu(\nu) = \frac{8\pi\nu^2}{c^3} \bar{E} = \frac{8\pi h\nu^3}{c^3} \frac{1}{\exp\left(\frac{h\nu}{k_B T}\right)-1}, \quad (57)$$

which is Planck's law!

The Stefan-Boltzmann law

The Stefan-Boltzmann law has been derived much earlier than the Planck distribution function, and it basically states which is the temperature dependence of radiation. It is going to be derived, with few simple steps, from (57).

HR27

Let U be defined as:

$$U = \int_0^\infty u_\nu(\nu)\, d\nu . \quad (58)$$

Then, it is possible to compute U explicitly substituting (57):

$$U = \int_0^\infty \frac{8\pi h \nu^3}{c^3} \frac{1}{\exp\left(\frac{h\nu}{K_B T}\right) - 1}\, d\nu \quad (59)$$

Let's apply the change of variables

$$x = \frac{h\nu}{K_B T}, \longleftrightarrow \nu = \frac{K_B T}{h} x$$

with differentials

$$dx = \frac{h}{K_B T} d\nu . \longleftrightarrow d\nu = \frac{K_B T}{h} dx .$$

We have:

$$U = \frac{8\pi h}{c^3} \left(\frac{K_B T}{h}\right)^3 \frac{K_B T}{h} \int_0^\infty \frac{x^3}{\exp(x) - 1}\, dx .$$

The integral is very complicated, but it is known that its result is

$$\int_0^\infty \frac{x^3}{\exp(x) - 1}\, dx = \frac{\pi^4}{15},$$

so that

$$U = \frac{8\pi^5 K_B^4}{15 (ch)^3} T^4 \quad (60)$$

Eq. (60) is the Stefan-Boltzmann law, and this, finally, gives some sense about the T^4 dependence of heat radiation!

Wien's law

Stefan-Boltzmann law is achieved by integrating (57). On the other hand, by differentiating (57) it is possible to achieve another interesting law, first proposed by Wien, describing how the peak of Planck's distribution shifts with temperature!

Wien's law can be implemented numerically according to the following idea. We want to find the position of the maximum of (57), recalled here:

$$u_\nu(\nu) = \frac{8\pi h\nu^3}{c^3} \frac{1}{\exp\left(\frac{h\nu}{k_BT}\right)-1} ;$$

so, we are looking for the zeros of its derivative:

$$\frac{\partial u_\nu}{\partial \nu} = 0 .$$

To this aim, we can compute

$$\frac{\partial u_\nu}{\partial \nu} = \frac{8\pi h\nu^3}{c^3} \frac{0-\frac{h}{k_BT}\exp\left(\frac{h\nu}{k_BT}\right)}{\left[\exp\left(\frac{h\nu}{k_BT}\right)-1\right]^2} + 3\frac{8\pi h\nu^2}{c^2} \frac{1}{\exp\left(\frac{h\nu}{k_BT}\right)-1} =$$

$$= \frac{8\pi h\nu^2}{c^3} \frac{1}{\exp\left(\frac{h\nu}{k_BT}\right)-1} \left[-\frac{h\nu}{k_BT} \frac{\exp\left(\frac{h\nu}{k_BT}\right)}{\exp\left(\frac{h\nu}{k_BT}\right)-1} + 3\right] = 0$$

Under the substitution

$$x = \frac{h\nu}{k_BT} ,$$

The zero of the derivative corresponds to the zero of the expression

$$-x \frac{\exp(x)}{\exp(x)-1} + 3 = 0 \qquad (61)$$

Do not try fancy things like multiplying both members times the denominator: you would add a zero in $x=0$, misleading the numerical solver!

What we have to do is to formulate a Newton's method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad (62)$$

where

$$f(x_n) = -x_n \frac{\exp(x_n)}{\exp(x_n)-1} + 3 \qquad (63)$$

$$f'(x_n) = -\frac{(x_n \exp(x_n) + \exp(x_n))(\exp(x_n)-1) - x_n \exp(x_n)\exp(x_n)}{(\exp(x_n)-1)^2}$$

$f(x_n)$ should be monotonic so the guess should not be important; try, e.g., $x_0 = 0.5$.

As a reference result, you should find, @ convergence,

$x_\infty = 2.8214$.

Notes:

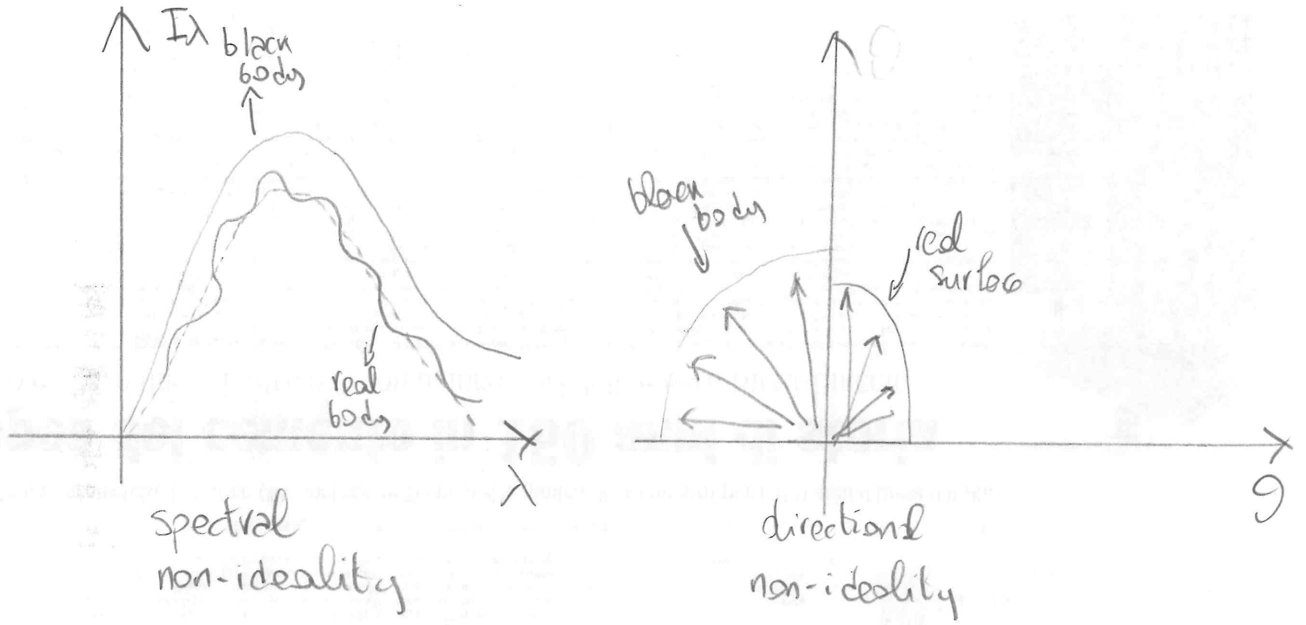
- This is valid for all temperatures: ν can be found inverting $x = \frac{h\nu}{k_B T}$
- This is <u>not</u> valid to find λ_{max}, because the distributions in λ and ν are different. In particular, $\nu_{max} \neq \frac{c}{\lambda_{max}}$: to find λ_{max} one should set the Newton's method for $u_\lambda(\lambda)$, instead of for $u_\nu(\nu)$!

Radiative properties of non-black surfaces

HR30

So far, the most relevant formulas we achieved are valid for an ideal black body. However real bodies are neither perfect absorbers, nor perfect emitters. In this view, the black body is kind of a "reference, ideal case" to which real bodies can be compared. In this view, the scope of this section is to introduce all the instruments aimed at quantifying this comparison.

There are two possible non-idealities (that could also manifest together):



spectral, and directional.

- About spectral, the λ dependence of $I_{\lambda,e}$ for a real body could be quite funny, as in the picture above. A possibility is to adopt gray body models (dashed), that are approximations of the spectral behaviour of the real body, designed in such a way that the ratio of emission at a particular wavelength to that of a black body is fixed:

$$\frac{I_{\lambda,g}(\lambda,T)}{I_{\lambda,b}(\lambda,T)} \neq f(\lambda) \qquad (64)$$

[this is the gray-body assumption]

- About directional non-idealities the right panels provide an idea. A black body emits isotropically; a real body might emit differently for different θ (or even φ) angles! As shown in this panel,

the red surface emits differently (arrows have different lengths at different ϑ) in ϑ, the black body emission is isotropic.
Just like we introduced the definition of gray body relatively to spectral non-idealities, here we can introduce a similar definition, for diffuse surfaces, i.e., surfaces such that

$$\frac{I_{\lambda,d}(\lambda,T,\vartheta,\varphi)}{I_{b,\lambda}(\lambda,T)} \neq f(\vartheta,\varphi) \quad (65)$$

[this is the diffuse surface assumption]

If it is possible to assume that our real body is simultaneously gray and diffuse, then the ratio of its intensity to the black body one does depend only on T: this is the gray-diffuse approximation.

Spectral directional emissivity $\epsilon_\lambda'(\lambda,T,\vartheta,\varphi)$

The spectral directional emissivity, indicated as $\epsilon_\lambda'(\lambda,T,\vartheta,\varphi)$, is defined as the ratio of the spectral directional intensity of emission of a real surface to that of a black body at the same λ and T:

$$\epsilon_\lambda'(\lambda,T,\vartheta,\varphi) = \frac{I_{\lambda,e}(\lambda,T,\vartheta,\varphi)}{I_{\lambda,b}(\lambda,T)}. \quad (66)$$

So: ϵ_λ' is dimensionless, as it involves a ratio of two emissivities, and < 1, since $I_{\lambda,b}$ is the highest achievable intensity. Eq. (66) is, therefore, basically a benchmark of the efficiency of emission of our real surface. However, if we re-write it as

$$I_{\lambda,e}(\lambda,T,\vartheta,\varphi) = \epsilon_\lambda'(\lambda,T,\vartheta,\varphi)\, I_{\lambda,b}(\lambda,T), \quad (67)$$

accepted that we have characterized ϵ_λ' in some way, then (67) becomes an operative formula to evaluate the radiation intensity emitted from the real surface of interest!

For a gray surface, (64) suggests that

$$\epsilon'_{\lambda,g} = \epsilon'_{\lambda g}(T,\vartheta,\varphi), \qquad (68)$$

i.e., it does not dependent on λ, which means that the λ dependence of our body is that of the black body, with no distorsion.

For a diffuse surface, (65) suggests that

$$\epsilon'_{\lambda,d} = \epsilon'_{\lambda,d}(\lambda,T), \qquad (69)$$

i.e., it does not depend on ϑ or φ.

For a gray-diffuse body, finally,

$$\epsilon'_{\lambda,g,d} = \epsilon'_{\lambda,g,d}(T). \qquad (70)$$

Notice that, even for the gray-diffuse body, (70) is still a spectral directional quantity, because it has been not subject of any λ or angular integration, which is what we are going to do in the following sections.

In fact, in (66)-(70), ϵ'_λ, (with the apex) remarks this property of being directional, and the subscript λ, remarking the spectral dependence.

Hemispherical spectral emissivity

The spectral directional emissivity ϵ'_λ is the quantity containing all the information. However, one could be interested in quantifying for each λ, the total radiation emitted by our surface. Recalling (17), we had, for a black body,

$$E_{\lambda,b}(\lambda,T) = \int_0^{2\pi}\int_0^{\frac{\pi}{2}} I_{\lambda,b}(\lambda,T)\cos\vartheta\sin\vartheta\, d\vartheta\, d\varphi \overset{(20)}{=} \pi\, I_{b,\lambda}(\lambda,T) \qquad (71)$$

HR33

So, (71) indicates a hemispherical spectral quantity, because it is still resolved in λ, but integrated over a hemisphere.
Similarly, (71) can be applied to a real emitter, leading to

$$E_{\lambda,e}(\lambda,T) = \int_0^{2\pi}\int_0^{\frac{\pi}{2}} I_{\lambda,e}(\lambda,T,\vartheta,\varphi)\cos\vartheta\sin\vartheta\, d\vartheta\, d\varphi \qquad (72)$$

Starting from (71) and (72), it is possible to define the "hemispherical spectral emissivity" as their ratio:

$$\epsilon_{\lambda,e}(\lambda,T) = \frac{E_{\lambda,e}(\lambda,T)}{E_{\lambda,b}(\lambda,T)} \qquad (73)$$

It is obvious that this expression is independent of ϑ and φ regardless of the directional properties of the real body, thanks to the integration in $d\Omega$. Moreover, (73) can be written more explicitly, by substituting (71) at the denominator and (72) and (67) at the numerator, leading to:

$$\epsilon_{\lambda,e}(\lambda,T) = \frac{\int_0^{2\pi}\int_0^{\frac{\pi}{2}} \epsilon'_{\lambda,e}(\lambda,T,\vartheta,\varphi)\, \cancel{I_{b,\lambda}(\lambda,T)}\cos\vartheta\sin\vartheta\, d\vartheta\, d\varphi}{\pi\, \cancel{I_{b,\lambda}(\lambda,T)}} =$$

$$= \frac{1}{\pi}\int_0^{2\pi}\int_0^{\frac{\pi}{2}} \epsilon'_{\lambda,e}(\lambda,T,\vartheta,\varphi)\cos\vartheta\sin\vartheta\, d\vartheta\, d\varphi \qquad (74)$$

where the simplification was possible because $I_{b,\lambda}(\lambda,T)$ is isotropic, i.e., does not depend on ϑ or φ.
Notice that the quantities in this section do not contain the apex, which indicated, in particular, the property of the quantity to be directional, i.e., angularly resolved.

Directional total emissivity

In the previous section we integrated out the angular dependence from the spectral directional emissivity ϵ'_λ. Now, starting from it, we are not going to "touch" the angular dependence, but just λ: this is going to indicate a "total" quantity, in the sense that, even if still directional, already accounting for all spectral (λ) contributions.

To this aim, we are going to need a bit of additional definitions. In fact, (17) is integrated in $d\Omega$, and one could recall that $d\Omega = \sin\vartheta \, d\vartheta \, d\varphi$.

So, one should define a spectral, directional emissive power $E'_{\lambda,e}$ for a real body as

$$E'_{\lambda,e}(\lambda, T, \vartheta, \varphi) = I_{\lambda,e}(\lambda, T, \vartheta, \varphi) \cos\vartheta \qquad (75)$$

If the intensity is a property of the emitter only, (75) contains also information about the receiver, i.e., $\cos\vartheta$: the angle between the normal vectors of emitting and receiving surfaces. Yet, this is still a spectral and directional quantity.

Starting from (75) one can define the directional total emitted power as

$$E'_e(T, \vartheta, \varphi) = \int_0^\infty E'_{\lambda,e}(\lambda, T, \vartheta, \varphi) \, d\lambda = \int_0^\infty I_{\lambda,e}(\lambda, T, \vartheta, \varphi) \cos\vartheta \, d\lambda. \qquad (76)$$

This allows to define the directional total emissivity, ϵ', as:

$$\epsilon'(T, \vartheta, \varphi) = \frac{E'_e(T, \vartheta, \varphi)}{E'_b(T, \vartheta)}. \qquad (77)$$

One could be puzzled in noticing directional black body quantities, but don't worry: $E_b'(T,\theta)$ indicates simply (HR35)

$$E_b'(T,\theta) = \int_0^\infty I_{\lambda,b}(\lambda,T)\cos\theta \, d\lambda, \quad (78)$$

so that, actually no contraduction with the isotropic nature of $I_{\lambda,b}$ is present!

It is possible to write a bit more explicitly (77), by substituting (76) and (78):

$$\epsilon'(T,\theta,\varphi) = \frac{\int_0^\infty I_{\lambda,e}(\lambda,T,\theta,\varphi)\cancel{\cos\theta}\, d\lambda}{\int_0^\infty I_{\lambda,b}(\lambda,T)\cancel{\cos\theta}\, d\lambda} = \frac{\int_0^\infty \epsilon_\lambda'(\lambda,T,\theta,\varphi)\, I_b(\lambda,T)\, d\lambda}{\frac{\sigma T^4}{\pi}}, \quad (79)$$

where $\cos\theta$ can be simplified because the integrals do not involve θ, but $I_{\lambda,b}$ can not. On the other hand, the denominator can be computed explicitly from the Stefan-Boltzmann law, stating that

$$E_b(T) = \sigma T^4, \quad (80)$$

where σ is computed in a similar way to (60). Be careful, however, that (60) is not computed on E_b or $I_{\lambda,b}$, but on u, so there is some different constant, but the way of reaching the result is almost the same. In particular, from [2005 Longair, p. 13], we have

$$\sigma = \frac{2\pi^5 k_B^4}{15 c^2 h^3} \approx 5.67 \times 10^{-8} \ W\, m^{-2}\, K^{-4}. \quad (81)$$

Finally, the $\frac{1}{\pi}$ comes from (20): $E_b = \pi I_b$.

Hemispherical total emissivity

In the previous sections we have integrated $\epsilon'_{\lambda,e}$ once w.r.t. $d\Omega$, once w.r.t. $d\lambda$; now, we are going to discuss the result of both integrations. This leads to the idea of emissive power of a body, such as that used in (20):

$$E_e(T) = \int_0^{\infty}\int_0^{2\pi}\int_0^{\frac{\pi}{2}} E'_{\lambda,e}(\lambda,T,\vartheta,\varphi)\sin\vartheta \, d\vartheta \, d\varphi \, d\lambda =$$

$$= \int_0^{\infty}\int_0^{2\pi}\int_0^{\frac{\pi}{2}} I_{\lambda,e}(\lambda,T,\vartheta,\varphi)\sin\vartheta\cos\vartheta \, d\vartheta \, d\varphi \, d\lambda =$$

$$= \int_0^{\infty}\int_0^{2\pi}\int_0^{\frac{\pi}{2}} \epsilon'_{\lambda,e}(\lambda,T,\vartheta,\varphi) \, I_{\lambda,b}(\lambda,T)\sin\vartheta\cos\vartheta \, d\vartheta \, d\varphi \, d\lambda, \quad (82)$$

where (20) instead describes $E_b(T)$: exactly the same expression, without $\epsilon'_{\lambda,e}$. This allows to define the hemispherical total emissivity, $\epsilon(T)$, as:

$$\epsilon(T) = \frac{E_e(T)}{E_b(T)}. \quad (83)$$

However, we have just mentioned that

$$E_b(T) = \sigma T^4,$$

so

$$\epsilon(T) = \frac{1}{\sigma T^4}\int_0^{\infty}\int_0^{2\pi}\int_0^{\frac{\pi}{2}} \epsilon'_{\lambda,e}(\lambda,T,\vartheta,\varphi) \, I_{\lambda,b}(\lambda,T)\sin\vartheta\cos\vartheta \, d\vartheta \, d\varphi \, d\lambda. \quad (84)$$

To conclude this section, let's consider a very specific case, i.e., that of a grey diffuse body. In this case, $\epsilon'_{\lambda,e} = \epsilon'_{\lambda,e}(T)$, so it can be factorized out of the integral, leading to

$$\epsilon(T) = \frac{1}{\sigma T^4} \epsilon'_{\lambda,e}(T) \int_0^{\infty} \int_0^{2\pi} \int_0^{\frac{\pi}{2}} I_{b,\lambda}(\lambda,T) \cos\theta \sin\theta \, d\theta \, d\varphi \, d\lambda =$$

$$= \frac{1}{\sigma T^4} \epsilon'_{\lambda,e}(T) \, E_b(T) = \frac{1}{\cancel{\sigma T^4}} \cancel{\sigma T^4} \epsilon'_{\lambda,e}(T). \quad (85)$$

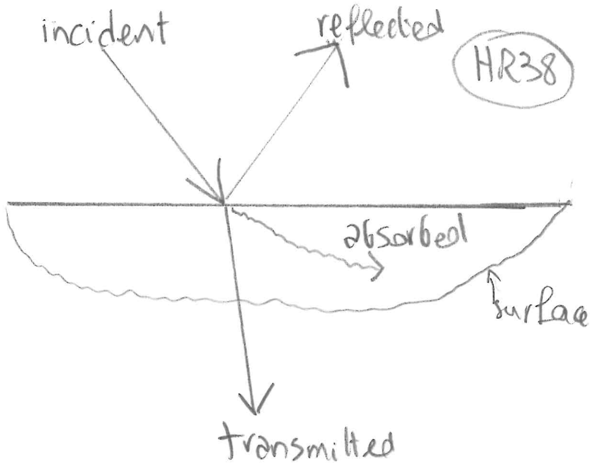
Therefore, if we have the spectral directional emissivity for a diffuse gray body then it is also the hemispherical total emissivity!

Absorptivity

In the previous section we focused on emissivity, i.e., the capability of a material to radiate. In other situations we are not just interested in emission of heat radiation, but also/especially in absorption of light. This is, for example, the case of solar cells, where we want to intercept (and absorb) the maximum amount of radiation that we can.

When dealing with absorption, we are considering a different perspective w.r.t. emissivity. In emissivity, we try to quantify how many radiation we generate due to the body's temperature. In absorption problems, light is already present, and we are studying how our body is reacting when such "external" rays are impinging on it.

This sketch attempts to summarize what happens to radiation. We start from an incident radiation, generated somehow and somewhere far from our body/surface. What may happen to our incident radiation?



- Part of it can be reflected by our surface.
- Part of it overcomes the surface and gets transmitted through it.
- Part of it is lost, absorbed by our body and converted into other forms of energy (heat, electricity, whatever ...).

To summarize,

$$Q_{inc} = Q_{refl} + Q_{trans} + Q_{abs}, \qquad (86)$$

[where Q has dimension of power, i.e., W, or of power density, i.e., $\frac{W}{m^2}$]

which becomes, after dividing both sides by Q_{inc},

$$\underbrace{\frac{Q_{inc}}{Q_{inc}}}_{1} = \underbrace{\frac{Q_{refl}}{Q_{inc}}}_{\rho} + \underbrace{\frac{Q_{trans}}{Q_{inc}}}_{\tau} + \underbrace{\frac{Q_{abs}}{Q_{inc}}}_{\alpha}, \qquad (87)$$

i.e.,

$$\rho + \tau + \alpha = 1.$$

ρ is referred to as "reflectivity"

τ is referred to as "transmissivity"

α is referred to as "absorptivity".

These definitions, so far, are unrelated to all the black body stuff that we have introduced in the previous sections. However, is it possible to make some connection with them? And, maybe, with the concept of emissivity? HR39

First, notice that (86) and (87) refer to hemispherical total quantities, but the very same expressions hold wavelength by wavelength, angle by angle, so (87) could be generalized as:

$$\alpha_\lambda + \rho_\lambda + \gamma_\lambda = 1 , \quad (88)$$

wavelength-by-wavelength, or, for smooth surfaces, even angle-by-angle:

$$\alpha_\lambda' + \rho_\lambda' + \gamma_\lambda' = 1 \quad (89)$$

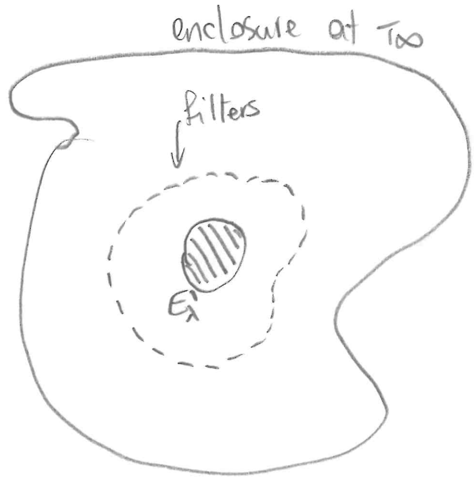
However, let's stay focused on the equation involving integrated quantities, (87). To approach the black body problem, let's assume that our surface is opaque. In this view, scattering kills the transmission through this object, and (87) reduces to

$$\alpha + \rho = 1 \quad (90)$$

or, equivalently,

$$\alpha = 1 - \rho , \quad (91)$$

which provides an operative way of computing the absorptivity.



Relating absorptivity to emissivity

Consider the following concept experiment:

an evacuated chamber at $T = T_\infty$. Inside, there is a body, whose spectral directional emissivity is $\epsilon_\lambda'(\lambda, T, \vartheta, \varphi)$, at an initial temperature T_w.

As usual, we have no conduction/convection due to vacuum, therefore only radiation might take place.

With respect to other experiments, the new entry is given by the filters, in particular band-pass filters allowing only a narrow λ range (e.g., $3.6 \div 3.7\ \mu m$)

to pass. Because of those filters, only radiation within this range can transport heat out of the body and towards the enclosure walls: any other λ contribution will be eventually back-reflected and re-absorbed into the body. Now, let's consider that transients are over, and we are at steady state, at equilibrium. Here, the net heat transfer is zero, but there is still some exchange between body and outer walls. Still, this exchange may involve only $\lambda \in [3.6, 3.7]$ μm. In other words, our body can emit/absorb only in this range. A similar consideration/experiment could be carried out for the direction $d\Omega$. So, just like we defined (15), recalled here:

$$I_{\lambda,e}(\lambda, T, \theta, \varphi) = \frac{dQ_{emit}}{dA_s \cos\theta \, d\Omega_o \, d\lambda},$$

we can define a spectral directional absorptivity as:

$$\alpha_\lambda'(\lambda, T, \theta, \varphi) = \frac{dQ_{abs}}{dA_i \cos\theta_i \, d\Omega_i \, d\lambda \, I_{\lambda,i}}, \quad (92)$$

where we also have $I_{\lambda,i}$ at the denominator, using an idea/expression similar to (75). The major difference of this definition w.r.t. those pertaining the emissivity lies in the fact that the quantities at the denominator concern incidence on our body, rather than emission from it.

On the basis of these preliminary definitions, we are now ready to relate α to ϵ. First, one could simply state that the net heat transfer from one body towards another results from the difference of the radiation outgoing from it, minus that impinging on it:

$$Q_{net} = Q_{outgoing} - Q_{incoming} \quad (93)$$

Concerning the incoming radiation, it is simply the radiation incident on our body:

$$Q_{incoming} = Q_{incident}. \quad (94)$$

About the outgoing radiation, the two "sources" are the emitted radiation, and the radiation reflected by our body:

$$Q_{outgoing} = Q_{emitted} + Q_{reflected} \,. \quad (95)$$

So,

$$Q_{net} = Q_{emitted} + Q_{reflected} - Q_{incident} \,. \quad (96)$$

Now, recall (86), with $Q_{trans} = 0$; then, we have

$$Q_{inc} - Q_{refl} = Q_{abs} \,, \quad (97)$$

so, in (96), we have:

$$Q_{net} = Q_{emitted} - Q_{absorbed} \,. \quad (98)$$

Now: at equilibrium, $Q_{net} = 0$, so

$$Q_{emitted} = Q_{absorbed} \quad (99).$$

Let's recall that, with our filters, we can select emission for just few θ, φ, λ. Then, we can recall (15) and (92), with some angles and λ, leading to:

$$\alpha(\lambda, T, \theta, \varphi)\, I_{\lambda,i}(\lambda, \theta_i, \varphi_i)\, \cancel{\cos\theta_i}\, \cancel{d\Omega_i}\, \cancel{d\lambda} = \epsilon_\lambda'(\lambda, T, \theta, \varphi)\, I_{b,\lambda}(\lambda, T)\, \cancel{\cos\theta_i}\, \cancel{d\Omega_i}\, \cancel{d\lambda}$$

If we assume that our enclosure (evacuated) behaves as a black body, which could be reasonable considering that our cavity is much larger than our body (hence radiation isotropic in it, also considering the perfect multiple reflections),

$$I_{\lambda,i} \approx I_{b,\lambda} \,,$$

and, ultimately,

$$\alpha_\lambda'(\lambda, T, \theta, \varphi) = \epsilon_\lambda'(\lambda, T, \theta, \varphi) \,,$$

which is Kirchhoff's law, stating that the spectral directional absorptivity is equal to the spectral directional emissivity: $\forall \lambda, \theta, \varphi, T$.

- Secondo me la definizione (75) andrebbe presentata prima di (17). HRERR
- La notazione con apici e subscript andrebbe presentata al punto precedente, comunque poco prima di (17).
- Preparare list of symbols: lista delle variabili e loro nome (e/o unità di misura)

1

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